

Cayley-type theorems for g -dimonoids

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Abstract. In this paper we prove Cayley-type theorems for g -dimonoids using the left (right) acts of sets and concept of dialgebra.

Key Words: g -dimonoid, dimonoid, act of set, dialgebra, morphism of acts, (l, r) -morphism of semigroup.

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Introduction

The concepts of a dimonoid and a dialgebra were introduced by Loday [1]. Dimonoids are a tool to study Leibniz algebras [1]. A dimonoid is a set with two binary associative operations satisfying certain additional identities. A dialgebra is a linear analogy of a dimonoid. The concept of a g -dimonoid (a generalized dimonoid) is introduced in [2]. The Cayley-type theorems for dimonoids are proved in [3].

In this paper two Cayley-type theorems for g -dimonoid are suggested.

Definition 1 An algebra $(X; \prec, \succ)$ with two binary operations is called g -dimonoid [2] if it satisfies the following four identities of associativity:

$$(x \prec y) \prec z = x \prec (y \prec z), \quad (A_1)$$

$$(x \prec y) \prec z = x \prec (y \succ z), \quad (A_2)$$

$$(x \prec y) \succ z = x \succ (y \succ z), \quad (A_3)$$

$$(x \succ y) \succ z = x \succ (y \succ z). \quad (A_4)$$

Definition 2 A g -dimonoid $(X; \prec, \succ)$ is called dimonoid [1] if it satisfies the following additional identity of associativity, too:

$$(x \succ y) \prec z = x \succ (y \prec z). \quad (A_5)$$

Let us give two examples of a dimonoid.

Example 1 Let $(X; \prec)$ be a zero semigroup, that is $x \prec y = 0$ for all $x, y \in X$, where 0 is a fixed element of the set X . For fixed elements a, b of the set X , where $a \neq 0, b \neq 0, a \neq b$, we define on X the operation \succ , assuming

$$x \succ y = \begin{cases} a, & x = y = b, \\ 0, & \text{otherwise,} \end{cases}$$

for all $x, y \in X$. It is easy to see that $(X; \prec, \succ)$ is a dimonoid.

Example 2 Let $X \neq \emptyset$. Define the following operations on the set X :

$$x \succ y = y,$$

$$x \prec y = x,$$

for all $x, y \in X$. Then the algebra $(X; \prec, \succ)$ is a dimonoid.

Let us give an example of a g -dimonoid which is not a dimonoid.

Example 3 Let X be an arbitrary nonempty set, $|X| \geq 2$, and let X^* be the set of all finite nonempty words in the alphabet X . Denote the first (respectively, the last) letter of a word $\omega \in X^*$ by $\omega^{(0)}$ (respectively, by $\omega^{(1)}$). Define the following operations \prec, \succ on X^* :

$$\omega \prec u = \omega^{(0)},$$

$$\omega \succ u = u^{(1)},$$

for all $\omega, u \in X^*$. It is easy to check that the binary algebra $(X^*; \prec, \succ)$ is a g -dimonoid, but is not a dimonoid.

Definition 3 Let S be a semigroup, $X \neq \emptyset$. The map $(\cdot) : S \times X \rightarrow X; (s, x) \mapsto s \cdot x$ is called a left S -act of X if the identity $(st) \cdot x = s \cdot (t \cdot x)$ holds.

Definition 4 Let S be a semigroup, $X \neq \emptyset$. The map $(\cdot) : X \times S \rightarrow X; (x, s) \mapsto x \cdot s$ is called a right S -act of X if the identity $x \cdot (st) = (x \cdot s) \cdot t$ holds.

Definition 5 Let the map (\cdot) be a left S -act of X and the map (\circ) be a left S -act of Y : The map $\varphi : X \rightarrow Y$ is called morphism of that left S -acts if $\varphi(s \cdot x) = s \circ \varphi(x)$ for all $x \in X$ and $s \in S$.

A morphism of right S -acts can be defined in a similar way. Let $(S; \bullet)$ be a semigroup, $X \neq \emptyset$. We can interpret the operation (\bullet) as a left and a right S -act of S .

Definition 6 Let $(S; \bullet)$ be a semigroup, $X \neq \emptyset$, and maps (\cdot) and (\circ) are respectively left and right S -acts of X ; then the map $\varphi : X \rightarrow S$ is called (l, r) -morphism of semigroup S if the following conditions are valid:

$$\varphi(s \cdot x) = s \bullet \varphi(x),$$

$$\varphi(x \circ s) = \varphi(x) \bullet s,$$

for all $x \in X$ and $s \in S$.

Definition 7 A dialgebra is a vector space over a field equipped with two binary bilinear operations satisfying the axioms of a dimonoid [1].

For the second order formulae (and the second order languages) see [4, 5, 6]. Let us recall, that a hyperidentity [7, 8, 9, 10, 11, 12] (or $\forall(\forall)$ -identity) is a second-order formula of the following form:

$$\forall X_1, \dots, X_m \forall x_1, \dots, x_n (\omega_1 = \omega_2),$$

where ω_1, ω_2 are words (terms) in the alphabet of functional variables X_1, \dots, X_m and objective variables x_1, \dots, x_n . However hyperidentities are usually presented without universal quantifiers: $\omega_1 = \omega_2$. A hyperidentity $\omega_1 = \omega_2$ is said to be satisfied in the algebra $(Q; \Sigma)$ if this equality holds whenever every object variable x_j is replaced by an arbitrary element from Q and every functional variable X_i is replaced by an arbitrary operation of the corresponding arity from Σ . A possibility of such replacement is assumed, that is:

$$\{|X_1|, \dots, |X_m|\} \subseteq \{|A| \mid A \in \Sigma\} = T_{(Q, \Sigma)} = T_{(\Sigma)},$$

where $|S|$ is the arity of S , and $T_{(Q, \Sigma)}$ is called the arithmetic type of $(Q; \Sigma)$. A T -algebra is an algebra with arithmetic type $T \subseteq N$. A class of algebras is called a class of T -algebras if every algebra in it is a T -algebra.

Definition 8 An algebra $(Q; \Sigma)$ is called idempotent, if the following hyperidentity of idempotency is valid:

$$X(\underbrace{x, \dots, x}_n) = x, \tag{id}$$

for all $n \in T_{(Q; \Sigma)}$.

The hyperidentity is said to be non-trivial if $m > 1$, and it is trivial if $m = 1$. The number m is called the functional rank of the given hyperidentity.

A binary algebra $(Q; \Sigma)$ is said to be a q -algebra (e -algebra) if there is an operation $A \in \Sigma$ such that $Q(A)$ is a quasigroup (a groupoid with a unit).

A binary algebra $(Q; \Sigma)$ is called non-trivial if $|\Sigma| > 1$. It is known [7, 8] (see also [9, 13]) that if an associative non-trivial hyperidentity is satisfied in a non-trivial q -algebra (e -algebra), then this hyperidentity can only be of the functional rank 2 and of one of the following forms:

$$X(x, Y(y, z)) = Y(X(x, y), z), \quad (asm)_1$$

$$X(x, Y(y, z)) = X(Y(x, y), z), \quad (asm)_2$$

$$Y(x, Y(y, z)) = X(X(x, y), z). \quad (asm)_3$$

Moreover, in the class of q -algebras (e -algebras) the hyperidentity $(asm)_3$ implies the hyperidentity $(asm)_2$ which, in its turn, implies the hyperidentity $(asm)_1$.

The algebra $(Q; \Sigma)$ is called hyperassociative, if it satisfies the hyperidentity of associativity $(asm)_1$.

Example 4 Let A, B be nonempty sets, Σ be the set of all mappings from B to A , and Q be the set of all mappings from A to B . Then every element $\alpha \in \Sigma$ can be considered as a binary operation on Q ,

$$\alpha(a, b) = a \cdot \alpha \cdot b,$$

where $a, b \in Q$ and $a \cdot \alpha \cdot b$ is the usual superposition of mappings. So we obtain hyperassociative algebra $(Q; \Sigma)$. Moreover, if $A = B$ we obtain second degree (order) algebra $(Q; \Sigma; \cdot)$ in the sense of [14].

Thus, hyperassociative algebras are algebras with semigroup operations. Hyperassociative algebras under the name of Γ -semigroups (gamma-semigroups) or doppelsemigroups were considered by various authors [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

Note that a g -dimonoid $(Q; \prec, \succ)$ is hyperassociative iff (A_5) and the following identity of associativity hold:

$$(x \prec y) \succ z = x \prec (y \succ z). \quad (A_6)$$

1 Main results

A Cayley-type theorem for dimonoids was proved in [3]. In this chapter we prove a more general result.

Theorem 1 Let S be a semigroup, $X \neq \emptyset$, and maps (\cdot) and (\circ) be left and right S -acts of X respectively, and let the map $\varphi : X \rightarrow S$ be a (l, r) -morphism of semigroup S . Define the operations \succ and \prec as follows:

$$\succ : X \times X \rightarrow X, \quad x \succ y := \varphi(x) \cdot y, \quad \forall x, y \in X,$$

$$\prec: X \times X \rightarrow X, \quad x \prec y := x \circ \varphi(y), \quad \forall x, y \in X.$$

Then the algebra $(X; \prec, \succ)$ is a g -dimonoid, which we denote by $C_S(X)$ and call Cayley's g -dimonoid.

Conversely, for any arbitrary g -dimonoid G there exists a semigroup S , and $X \neq \emptyset$, and left and right S -acts of X , and an (l, r) -morphism of semigroup S such that the g -dimonoid G coincides with $C_S(X)$, i.e. $G = C_S(X)$.

Proof. Let us prove the first part of the result. For this, it suffices to verify the following four identities:

$$(A_1) \quad (x \prec y) \prec z = (x \circ \varphi(y)) \prec z = (x \circ \varphi(y)) \circ \varphi(z) = x \circ (\varphi(y)\varphi(z)) = x \circ \varphi(y \circ \varphi(z)) = x \prec (y \circ \varphi(z)) = x \prec (y \prec z),$$

$$(A_2) \quad (x \prec y) \prec z = (x \prec y) \circ \varphi(z) = (x \circ \varphi(y)) \circ \varphi(z) = x \circ (\varphi(y)\varphi(z)) = x \circ \varphi(\varphi(y) \cdot z) = x \prec (\varphi(y) \cdot z) = x \prec (y \succ z),$$

$$(A_3) \quad (x \prec y) \succ z = (x \circ \varphi(y)) \succ z = \varphi(x \circ \varphi(y)) \cdot z = (\varphi(x)\varphi(y)) \cdot z = \varphi(x) \cdot (\varphi(y) \cdot z) = x \succ (\varphi(y) \cdot z) = x \succ (y \succ z),$$

$$(A_4) \quad (x \succ y) \succ z = (\varphi(x) \cdot y) \succ z = \varphi(\varphi(x) \cdot y) \cdot z = (\varphi(x)\varphi(y)) \cdot z = \varphi(x) \cdot (\varphi(y) \cdot z) = x \succ (\varphi(y) \cdot z) = x \succ (y \succ z).$$

Hence, the algebra $(X; \prec, \succ)$ is a g -dimonoid and $(X; \prec, \succ) = C_S(X)$.

Now we prove the second part of the result. Let the algebra $D = (D; \prec, \succ)$ be a g -dimonoid. We denote by \sim the least congruence on D for which the quotient g -dimonoid is a semigroup. It is clear that \sim is the congruence generated by the relation $\sigma = \{(a, a), (a \succ b, a \prec b), (a \prec b, a \succ b) : a, b \in D\}$. Let $x \sim y$. According to the definition of the congruence, we can say that there exist $b_1, c_1, \dots, b_n, c_n \in D$ and $o_1, o'_1, \dots, o_n, o'_n \in \{\succ, \prec\}$ such that the following condition holds:

$$x = (b_1 o_1 c_1) \sigma (b_1 o'_1 c_1) = (b_2 o_2 c_1) \sigma (b_2 o'_2 c_2) = \dots = (b_n o_n c_n) \sigma (b_n o'_n c_n) = y.$$

According to the identities A_3, A_4 , we have:

$$(b_i o_i c_i) \succ a = (b_i o'_i c_i) \succ a, \quad \forall i = 1, \dots, n, \quad \forall a \in D.$$

Thus,

$$\begin{aligned} x \succ a &= (b_1 o_1 c_1) \succ a = (b_1 o'_1 c_1) \succ a = \dots = (b_n o_n c_n) \succ a = \\ &= (b_n o'_n c_n) \succ a = y \succ a \rightarrow x \succ a = y \succ a, \quad \forall a \in D. \end{aligned}$$

According to the identities A_1, A_2 , we have $a \prec x = a \prec y, \forall a \in D$, i.e.

$$x \sim y \rightarrow x \succ a = y \succ a \& a \prec x = a \prec y, \quad \forall a \in D.$$

It means, that the congruence \sim is contained in the congruence:

$$\tau = \{(x, y) \in D \times D \mid x \succ a = y \succ a \text{ \& } a \prec x = a \prec y, \forall a \in D\}.$$

Now let us pick any congruence θ which satisfies the condition $\sim \subseteq \theta \subseteq \tau$. Hence D/θ is an algebra with two binary operations, which are equal, because θ contains the congruence \sim . So D/θ is a semigroup, which we denote by S , and for arbitrary $x \in D$ we denote by $[x]$, the class of the congruence θ that contains x . Define the maps:

$$\cdot : S \times D \rightarrow D, [x] \cdot y := x \succ y, \quad (1)$$

$$\circ : D \times S \rightarrow D, x \circ [y] := x \prec y. \quad (2)$$

The correctness of definitions of the maps \cdot and \circ follows from the fact that $\theta \subseteq \tau$. We now check that the maps \cdot and \circ are left and right S -acts of D , respectively:

$$\begin{aligned} ([a][b]) \cdot x &= [a \prec b] \cdot x \stackrel{[1]}{=} (a \prec b) \succ x \stackrel{(A_3)}{=} \\ a \succ (b \succ x) &\stackrel{[1]}{=} a \succ ([b] \cdot x) \stackrel{[1]}{=} [a] \cdot ([b] \cdot x), \end{aligned}$$

$$\begin{aligned} x \circ ([a][b]) &= x \circ [a \succ b] \stackrel{[2]}{=} x \prec (a \succ b) \stackrel{(A_2)}{=} \\ (x \prec a) \prec b &\stackrel{[2]}{=} (x \circ [a]) \prec b \stackrel{[2]}{=} (x \circ [a]) \circ [b]. \end{aligned}$$

We now consider the map $\Phi : D \rightarrow S, \Phi(x) = [x]$ and show that it is a (l, r) -morphism of S :

$$\Phi([a] \cdot x) = \Phi(a \succ x) = [a \succ x] = [a][x] = [a]\Phi(x),$$

$$\Phi(x \circ [a]) = \Phi(x \prec a) = [x \prec a] = [x][a] = \Phi(x)[a].$$

Note, that the conditions of the first part of the theorem are satisfied, so $(D; \prec', \succ') = C_S(D)$ where:

$$\succ' : D \times D \rightarrow D, x \succ' y := \Phi(x) \cdot y, \quad \forall x, y \in D,$$

$$\prec' : D \times D \rightarrow D, x \prec' y := x \circ \Phi(y), \quad \forall x, y \in D.$$

Also note that:

$$\succ' = \succ,$$

$$\prec' = \prec,$$

hence $(D; \prec', \succ') = C_S(D)$. \square

Let K be a field, X be a nonempty set: $X = \{x_i \mid i \in I\}$. Denote $G := \{\{\alpha_i x_i \mid i \in I\} \mid \exists J \subset I, |I \setminus J| < \infty, \alpha_j = 0, \forall j \in J, \alpha_i \in K\}$, where $\alpha_i x_i$ is just symbol. For any $x_t \in X$, considering $x_t = \{\alpha_i x_i \mid i \in I\}$, where $\alpha_t = 1$, and $\alpha_i = 0$ if $i \neq t$, we obtain that $X \subseteq G$. Now we define the operation $+$ on G :

$$\{\alpha_i x_i \mid i \in I\} + \{\beta_i x_i \mid i \in I\} = \{(\alpha_i + \beta_i) x_i \mid i \in I\}.$$

Because $K(+)$ is an abelian group, $G(+)$ will be an abelian group, too. We now define the operation $\cdot : K \times G \rightarrow G$ as follows:

$$\alpha \cdot \{\alpha_i x_i \mid i \in I\} = \{(\alpha \alpha_i) x_i \mid i \in I\}.$$

Then $G(+)$ with operation \cdot will be a free K -module, which we denote by $K[X]$. It is easy to see, that if D is a dimonoid, then $K[D]$ is a dialgebra. This fact is used in the proof of the following result.

Theorem 2 *Let S be a semigroup satisfying the following additional identity:*

$$xyz = tlp. \quad (*)$$

Define the operations \succ and \prec on $S \times S$ in the following way:

$$(g, h) \prec (k, l) = (gk, hk), \quad (op_1)$$

$$(g, h) \succ (k, l) = (gk, gl), \quad (op_2)$$

where $(g, h), (k, l) \in S \times S$. Then $\bar{S} = (S \times S; \prec, \succ)$ is a g -dimonoid, which satisfies the following hyperidentity:

$$X(x, Y(y, z)) = Z(T(t, l), p). \quad (**)$$

In particular the g -dimonoid \bar{S} will be hyperassociative.

*Conversely, for any g -dimonoid D with the hyperidentity $(**)$ there exists a semigroup H with the identity $(*)$ such that D is isomorphically embedded into \bar{H} .*

Proof. For proving the first part of the assertion, we use the previous theorem. Namely, we define the left and right S -acts of $S \times S$:

$$a \cdot (b, c) = (ab, ac),$$

$$(b, c) \circ a = (ba, ca).$$

We also define the operation $\varphi : S \times S \rightarrow S, (a, b) \mapsto a$ and check that the map φ is a (l, r) -morphism of S :

$$\varphi(a \cdot (b, c)) = \varphi((ab, ac)) = ab = a\varphi((b, c)),$$

$$\varphi((b, c) \circ a) = \varphi((ba, ca)) = ba = \varphi((b, c))a.$$

Then

$$(g, h) \prec (k, l) = (gk, hk) = (g, h) \circ k = (g, h) \circ \varphi(k, l),$$

$$(g, h) \succ (k, l) = (gk, gl) = g \cdot (k, l) = \varphi(g, h) \cdot (k, l),$$

for all $(g, h), (k, l) \in S \times S$. Now according to Theorem [1](#) we obtain, that \bar{S} is a g -dimonoid. Using the identity $(*)$, it is easy to check that \bar{S} satisfies the hyperidentity $(**)$. We should check the following cases:

1. $(g, h) \prec ((k, l) \succ (p, t)) = ((m, n) \prec (r, s)) \succ (u, v),$
2. $(g, h) \prec ((k, l) \succ (p, t)) = ((m, n) \prec (r, s)) \prec (u, v),$
3. $(g, h) \prec ((k, l) \succ (p, t)) = ((m, n) \succ (r, s)) \succ (u, v),$
4. $(g, h) \prec ((k, l) \succ (p, t)) = ((m, n) \succ (r, s)) \prec (u, v),$
5. $(g, h) \succ ((k, l) \succ (p, t)) = ((m, n) \prec (r, s)) \succ (u, v),$
6. $(g, h) \succ ((k, l) \succ (p, t)) = ((m, n) \prec (r, s)) \prec (u, v),$
7. $(g, h) \succ ((k, l) \succ (p, t)) = ((m, n) \succ (r, s)) \succ (u, v),$
8. $(g, h) \succ ((k, l) \succ (p, t)) = ((m, n) \succ (r, s)) \prec (u, v),$
9. $(g, h) \prec ((k, l) \prec (p, t)) = ((m, n) \prec (r, s)) \succ (u, v),$
10. $(g, h) \prec ((k, l) \prec (p, t)) = ((m, n) \prec (r, s)) \prec (u, v),$
11. $(g, h) \prec ((k, l) \prec (p, t)) = ((m, n) \succ (r, s)) \succ (u, v),$
12. $(g, h) \prec ((k, l) \prec (p, t)) = ((m, n) \succ (r, s)) \prec (u, v),$
13. $(g, h) \succ ((k, l) \prec (p, t)) = ((m, n) \prec (r, s)) \succ (u, v),$
14. $(g, h) \succ ((k, l) \prec (p, t)) = ((m, n) \prec (r, s)) \prec (u, v),$
15. $(g, h) \succ ((k, l) \prec (p, t)) = ((m, n) \succ (r, s)) \succ (u, v),$
16. $(g, h) \succ ((k, l) \prec (p, t)) = ((m, n) \succ (r, s)) \prec (u, v).$

Indeed:

1.

$$\begin{aligned} (g, h) \prec ((k, l) \succ (p, t)) &= (g, h) \prec (kp, kt) = (gkp, hkp) \stackrel{(*)}{=} \\ (mru, mrv) &= (mr, nr) \succ (u, v) = ((m, n) \prec (r, s)) \succ (u, v); \end{aligned}$$

2.

$$(g, h) \prec ((k, l) \succ (p, t)) = (g, h) \prec (kp, kt) = (gkp, hkp) \stackrel{(*)}{=} \\ (mru, nru) = (mr, nr) \prec (u, v) = ((m, n) \prec (r, s)) \prec (u, v);$$

3.

$$(g, h) \prec ((k, l) \succ (p, t)) = (g, h) \prec (kp, kt) = (gkp, hkp) \stackrel{(*)}{=} \\ (mru, mrv) = (mr, ms) \succ (u, v) = ((m, n) \succ (r, s)) \succ (u, v);$$

4.

$$(g, h) \prec ((k, l) \succ (p, t)) = (g, h) \prec (kp, kt) = (gkp, hkp) \stackrel{(*)}{=} \\ (mru, msu) = (mr, ms) \prec (u, v) = ((m, n) \succ (r, s)) \prec (u, v);$$

5.

$$(g, h) \succ ((k, l) \succ (p, t)) = (g, h) \succ (kp, kt) = (gkp, gkt) \stackrel{(*)}{=} \\ (mru, mrv) = (mr, nr) \succ (u, v) = ((m, n) \prec (r, s)) \succ (u, v);$$

6.

$$(g, h) \succ ((k, l) \succ (p, t)) = (g, h) \succ (kp, kt) = (gkp, gkt) \stackrel{(*)}{=} \\ (mru, nru) = (mr, nr) \prec (u, v) = ((m, n) \prec (r, s)) \prec (u, v);$$

7.

$$(g, h) \succ ((k, l) \succ (p, t)) = (g, h) \succ (kp, kt) = (gkp, gkt) \stackrel{(*)}{=} \\ (mru, mrv) = (mr, ms) \succ (u, v) = ((m, n) \succ (r, s)) \succ (u, v);$$

8.

$$(g, h) \succ ((k, l) \succ (p, t)) = (g, h) \succ (kp, kt) = (gkp, gkt) \stackrel{(*)}{=} \\ (mru, msu) = (mr, ms) \prec (u, v) = ((m, n) \succ (r, s)) \prec (u, v);$$

9.

$$(g, h) \prec ((k, l) \prec (p, t)) = (g, h) \prec (kp, lp) = (gkp, hkp) \stackrel{(*)}{=} \\ (mru, mrv) = (mr, nr) \succ (u, v) = ((m, n) \prec (r, s)) \succ (u, v);$$

10.

$$(g, h) \prec ((k, l) \prec (p, t)) = (g, h) \prec (kp, lp) = (gkp, hkp) \stackrel{(*)}{=} \\ (mru, nru) = (mr, nr) \prec (u, v) = ((m, n) \prec (r, s)) \prec (u, v);$$

11.

$$(g, h) \prec ((k, l) \prec (p, t)) = (g, h) \prec (kp, lp) = (gkp, hkp) \stackrel{(*)}{=} \\ (mru, mrv) = (mr, ms) \succ (u, v) = ((m, n) \succ (r, s)) \succ (u, v);$$

12.

$$(g, h) \prec ((k, l) \prec (p, t)) = (g, h) \prec (kp, lp) = (gkp, hkp) \stackrel{(*)}{=} \\ (mru, msu) = (mr, ms) \prec (u, v) = ((m, n) \succ (r, s)) \prec (u, v);$$

13.

$$(g, h) \succ ((k, l) \prec (p, t)) = (g, h) \succ (kp, lp) = (gkp, glp) \stackrel{(*)}{=} \\ (mru, mrv) = (mr, nr) \succ (u, v) = ((m, n) \prec (r, s)) \succ (u, v);$$

14.

$$(g, h) \succ ((k, l) \prec (p, t)) = (g, h) \succ (kp, lp) = (gkp, glp) \stackrel{(*)}{=} \\ (mru, nru) = (mr, nr) \prec (u, v) = ((m, n) \prec (r, s)) \prec (u, v);$$

15.

$$(g, h) \succ ((k, l) \prec (p, t)) = (g, h) \succ (kp, lp) = (gkp, glp) \stackrel{(*)}{=} \\ (mru, mrv) = (mr, ms) \succ (u, v) = ((m, n) \succ (r, s)) \succ (u, v);$$

16.

$$(g, h) \succ ((k, l) \prec (p, t)) = (g, h) \succ (kp, lp) = (gkp, glp) \stackrel{(*)}{=} \\ (mru, msu) = (mr, ms) \prec (u, v) = ((m, n) \succ (r, s)) \prec (u, v).$$

Now, we turn to the proof of the second part of the theorem. Let $D = (D; \prec, \succ)$ be a g -dimonoid satisfying the hyperidentity $(**)$. By D^0 we denote D with an adjoined element 0 (called zero) such that $x*0 = 0 = 0*x$, for all $x \in D^0$, and $*$ $\in \{\succ, \prec\}$. Let K be a field and $H = (D/\theta)^0 \times K[D^0]$ (we invoke the notation from the proof of Theorem [11](#)). On H we define a binary operation \bullet :

$$([a], x) \bullet ([b], y) = ([a \succ b], (a \succ y) + (x \prec b)),$$

for all $a, b, x, y \in D^0$. This operation is well-defined, because for a congruence θ we have:

$$a\theta a' \rightarrow a \succ y = a' \succ y \ \& \ x \prec a = x \prec a',$$

for all $x, y \in D$.

We now check, that \bullet is associative:

$$\begin{aligned} (([a], x) \bullet ([b], y)) \bullet ([c], z) &= ([a \succ b], (a \succ y) + (x \prec b)) \bullet ([c], z) = \\ &= ([a \succ b \succ c], (a \succ b \succ z) + (((a \succ y) + (x \prec b)) \prec c)) = \\ &= ([a \succ b \succ c], (a \succ b \succ z) + ((a \succ y) \prec c) + (x \prec b \prec c)) \stackrel{(A_5)}{=} \\ &= ([a \succ b \succ c], (a \succ b \succ z) + (a \succ (y \prec c)) + (x \prec b \prec c)) = \\ &= ([a \succ b \succ c], (a \succ ((b \succ z) + (y \prec c))) + (x \prec b \prec c)) = \\ &= ([a], x) \bullet ([b \succ c], (b \succ z) + (y \prec c)) = \\ &= ([a], x) \bullet (([b], y) \bullet ([c], z)). \end{aligned}$$

Then $H(\bullet)$ is a semigroup with zero element $([0], 0)$. Define a map $f : D \rightarrow \overline{H}$ in the following way:

$$f(x) = ([x], 0), ([0], x).$$

It is easy to see, that f is an injective map. Furthermore:

$$\begin{aligned} f(a) \succ f(b) &= ([a], 0), ([0], a) \succ ([b], 0), ([0], b) = \\ &= ([a], 0) \bullet ([b], 0), ([a], 0) \bullet ([0], b) = \\ &= ([a \succ b], (a \succ 0) + (0 \prec b)), ([0], (a \succ b) + (0 \prec 0)) = \\ &= ([a \succ b], 0), ([0], a \succ b) = f(a \succ b). \end{aligned}$$

We obtain $f(a) \prec f(b) = f(a \prec b)$ in a similar way. Hence, f is a monomorphism of the g -dimonoids. Then D is isomorphically embedded into \overline{H} . Using the hyperidentity (**), it is easy to show that the semigroup \overline{H} satisfies the identity (*). Indeed:

$$\begin{aligned} ([g], x_g) \bullet ([k], x_k) \bullet ([t], x_t) &= ([g \succ k], (g \succ x_k) + (x_g \prec k)) \bullet ([t], x_t) = \\ &= ([g \succ k \succ t], (g \succ k \succ x_t) + ((g \succ x_k) \prec t) + ((x_g \prec k) \prec t)) \stackrel{(**)}{=} \\ &= ([h \succ r \succ l], (h \succ r \succ x_l) + ((h \succ x_r) \prec l) + ((x_h \prec r) \succ l)) = \\ &= ([h \succ r], (h \succ x_r) + (x_h \prec r)) \bullet ([l], x_l) = ([h], x_h) \bullet ([r], x_r) \bullet ([l], x_l). \end{aligned}$$

□

2 Corollaries

Corollary 1 ([3]) Let S be a semigroup and X be both a left S -act and a right S -act with commuting actions, and let $\varphi : X \rightarrow S$ be a (l, r) -morphism of S . Then the set X with two binary operations \prec and \succ defined by the following rules:

$$x \prec y := x \circ \varphi(y),$$

$$x \succ y := \varphi(x) \cdot y,$$

for all $x, y \in X$, is a dimonoid. Conversely, any dimonoid can be so constructed.

Corollary 2 (Cayley-type theorem for idempotent g -dimonoids) Let S be a semigroup and $X \neq \emptyset$, let the maps (\cdot) and (\circ) be left and right S -acts respectively, let the map $\varphi : X \rightarrow S$ be a (l, r) -morphism of S which satisfies the following condition:

$$\varphi(x) \cdot x = x \circ \varphi(x) = x, \forall x \in X. \quad (\#)$$

In this case $C_S(X)$ is an idempotent g -dimonoid.

Conversely, for any arbitrary idempotent g -dimonoid G there exists a semigroup S , and $X \neq \emptyset$, a left and a right S -acts of X , and a (l, r) -morphism of S , which satisfies $(\#)$ such that G coincides with $C_S(X)$, i.e. $G = C_S(X)$.

Corollary 3 (Cayley-type theorem for hyperassociative g -dimonoids) Let S be a semigroup and $X \neq \emptyset$, let the maps (\cdot) and (\circ) be respectively left and right S -acts, let the map $\Phi : X \rightarrow S$ be a (l, r) -morphism of S which satisfies the following conditions:

$$(\varphi(x)\varphi(y)) \cdot z = x \circ (\varphi(y)\varphi(z)), \quad \forall x, y, z \in X, \quad (\#\#)$$

$$(\varphi(x) \cdot y) \circ \varphi(z) = \varphi(x) \cdot (y \circ \varphi(z)), \quad \forall x, y, z \in X. \quad (\#\#\#)$$

In this case $C_S(X)$ is a hyperassociative g -dimonoid.

Conversely, for any arbitrary hyperassociative g -dimonoid G there exists a semigroup S , a non empty set X , and a left and a right S -acts of X , and a (l, r) -morphism of S , which satisfies $(\#\#)$ and $(\#\#\#)$, that G coincides with $C_S(X)$, i.e. $G = C_S(X)$.

Corollary 4 Let S be a commutative semigroup. Define the operations \succ and \prec on $S \times S$ as in (op_1) and (op_2) . Then $\bar{S} = (S \times S; \prec, \succ)$ is a g -dimonoid, which satisfies the following identity:

$$x \succ y = y \prec x. \quad (***)$$

Conversely, for any g -dimonoid D with the condition $(***)$ there exists a commutative semigroup H such that D is isomorphically embedded into \bar{H} .

Definition 9 A g -dialgebra is a vector space over a field equipped with two binary bilinear operations satisfying the axioms of a g -dimonoid.

Open problem 1. Prove a Cayley-type theorem for g -dialgebras.

Open problem 2. Characterize the free g -dialgebras.

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